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Chapter 12: Motion of Rigid bodies

Topic: Euler's equation of motion, torque free motion

Euler's Equations

Let us consider rotational system in which torque acting on a body is equal to rate of change of angular momentum. i.e.

$$\frac{dL}{dt} = N \quad \text{——} \quad (1)$$

where, L – Angular momentum

N – Torque

In a co-ordinate system rotating with the body, we have following relation,

$$\left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} + \vec{\omega}\times \quad \text{——} \quad (2)$$

In terms of body set (body axes),

$$\left(\frac{dL}{dt}\right)_{body} + \vec{\omega}\times\vec{L} = \vec{N} \quad \text{——} \quad (3)$$

The angular momentum can be written as,

$$\vec{L} = I_1 \omega_x \hat{i} + I_2 \omega_y \hat{j} + I_3 \omega_z \hat{k} \quad \text{——} \quad (4)$$

Now,

$$\left(\frac{dL}{dt}\right)_{body} = I_1 \dot{\omega}_x \hat{i} + I_2 \dot{\omega}_y \hat{j} + I_3 \dot{\omega}_z \hat{k}$$

Putting value of $\left(\frac{dL}{dt}\right)_{body}$ in eq^n ③.

$$\left(I_1 \dot{\omega}_x \hat{i} + I_2 \dot{\omega}_y \hat{j} + I_3 \dot{\omega}_z \hat{k}\right) + \vec{\omega} \times \vec{L} = \vec{N} \quad — (5)$$

where, $$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ I_1\omega_x & I_2\omega_y & I_3\omega_z \end{vmatrix}$$

$$\vec{\omega} \times \vec{L} = \hat{i}\left(\omega_y\omega_z I_3 - \omega_y\omega_z I_2\right) - \hat{j}\left(\omega_x\omega_z I_3 - \omega_x\omega_z I_1\right) + \hat{k}\left(\omega_x\omega_y I_2 - \omega_x\omega_y I_1\right)$$

Taking X-component of eq^n(5).

$$N_x = I_1 \dot{\omega}_x + \omega_y \omega_z (I_3 - I_2)$$

$$N_x = I_1 \dot{\omega}_x - \omega_y \omega_z (I_2 - I_3)$$

Similarly Y- & Z- components can be written as,

$$N_y = I_2 \dot{\omega}_y - \omega_z \omega_x (I_3 - I_1)$$

Note: $\frac{d}{dt} = \frac{d'}{dt} + \vec{\omega} \times$

& 6 $N_z = I_3 \dot{\omega}_z - \omega_x \omega_y (I_1 - I_2)$

For torque free motion i.e. $\vec{N} = 0$

$\Rightarrow N_x = N_y = N_z = 0$

$$I_1 \dot{\omega}_x = \omega_y \omega_z (I_2 - I_3) \quad \text{— ⑥}$$

$$I_2 \dot{\omega}_y = \omega_z \omega_x (I_3 - I_1) \quad \text{— ⑦}$$

$$I_3 \dot{\omega}_z = \omega_x \omega_y (I_1 - I_2) \quad \text{— ⑧}$$

Conservation of Kinetic Energy

Multiplying eq^n ⑥ by ω_x, ⑦ by ω_y & ⑧ by ω_z & then adding,

$$I_1 \omega_x \dot{\omega}_x + I_2 \omega_y \dot{\omega}_y + I_3 \omega_z \dot{\omega}_z = \omega_x \omega_y \omega_z (0)$$

$$\frac{d}{dt}\left(\frac{1}{2} I_1 \omega_x^2\right) + \frac{d}{dt}\left(\frac{1}{2} I_2 \omega_y^2\right) + \frac{d}{dt}\left(\frac{1}{2} I_3 \omega_z^2\right) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2} I_1 \omega_x^2 + \frac{1}{2} I_2 \omega_y^2 + \frac{1}{2} I_3 \omega_z^2\right) = 0$$

$$\Rightarrow \boxed{T = \frac{1}{2} I_1 \omega_x^2 + \frac{1}{2} I_2 \omega_y^2 + \frac{1}{2} I_3 \omega_z^2 = \text{constant}}$$

Conservation of Angular momentum

We know, $N = \dfrac{dL}{dt}$

For torque free motion $N = 0$

$$\Rightarrow \frac{dL}{dt} = 0$$

$$\Rightarrow L = \text{constant}$$

Euler's Angles

Three independent parameters which would completely specify the orientation of rigid body are called Euler's Angle.

Let x, y, z be the orthogonal space set of axis with unit vectors $\hat{i}, \hat{j}, \hat{k}$ along these axes.
Also x', y', z' be the orthogonal body set of axes with unit vectors $\hat{i}', \hat{j}', \hat{k}'$ along these axes.
Here the x-y plane is different from x'-y' plane and so on.
In order to account for rotatory motion we shall carry transformation from space set of axes to body set of axes.

The transformations is worked out through three successive rotation performed in some specific order. i.e. we rotate space set of axes x, y, z so as to coincide with body set of axes x', y', z' through three successive rotatory operations.