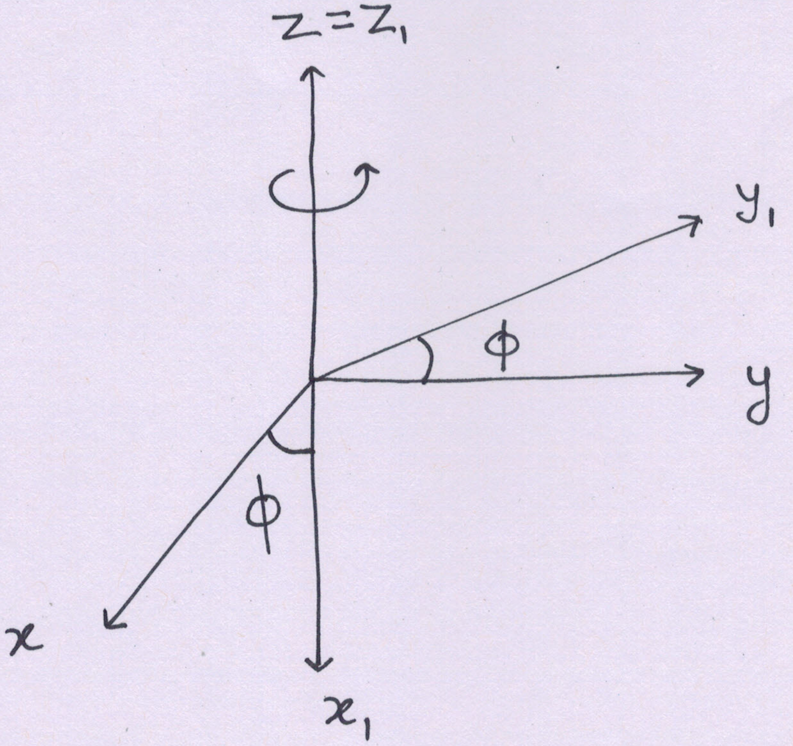
Euler's Angles

First Rotation

In this rotation, co-ordinate axes x, y and z are rotated anticlockwise by an angle ϕ about the z-axis, resulting the co-ordinate axis (x_1, y_1, z_1).



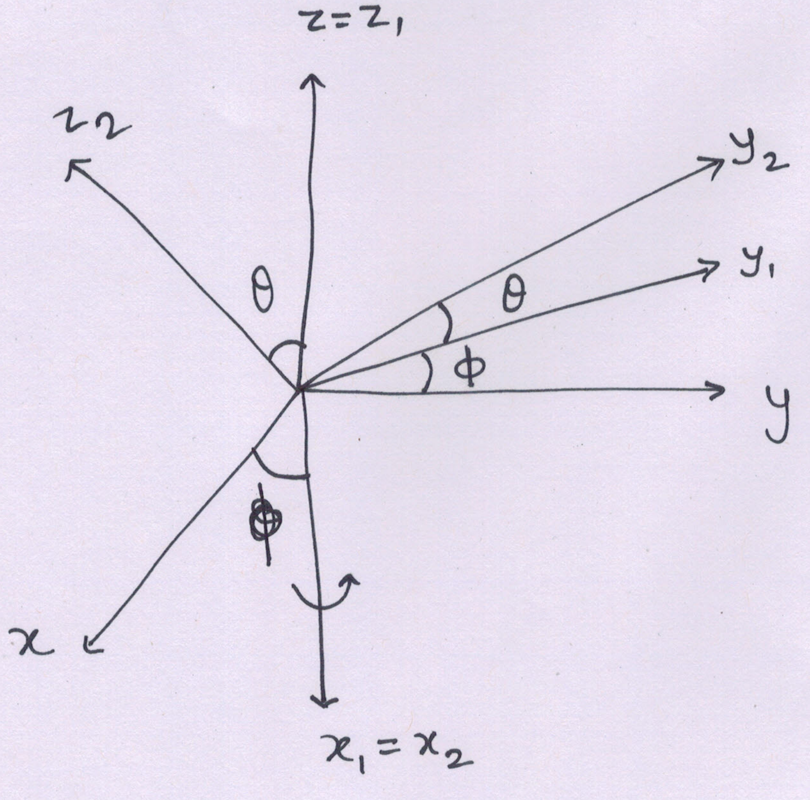
The transformation eq^n is,

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = D \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Second Rotation

In this rotation, co-ordinate axes x_1, y_1, z_1 are rotated about x_1 in anticlockwise direction by an angle θ. Then the resulting co-ordinate axes are, (x_2, y_2, z_2)

The line at which x_2 axis lies is called lines of node.



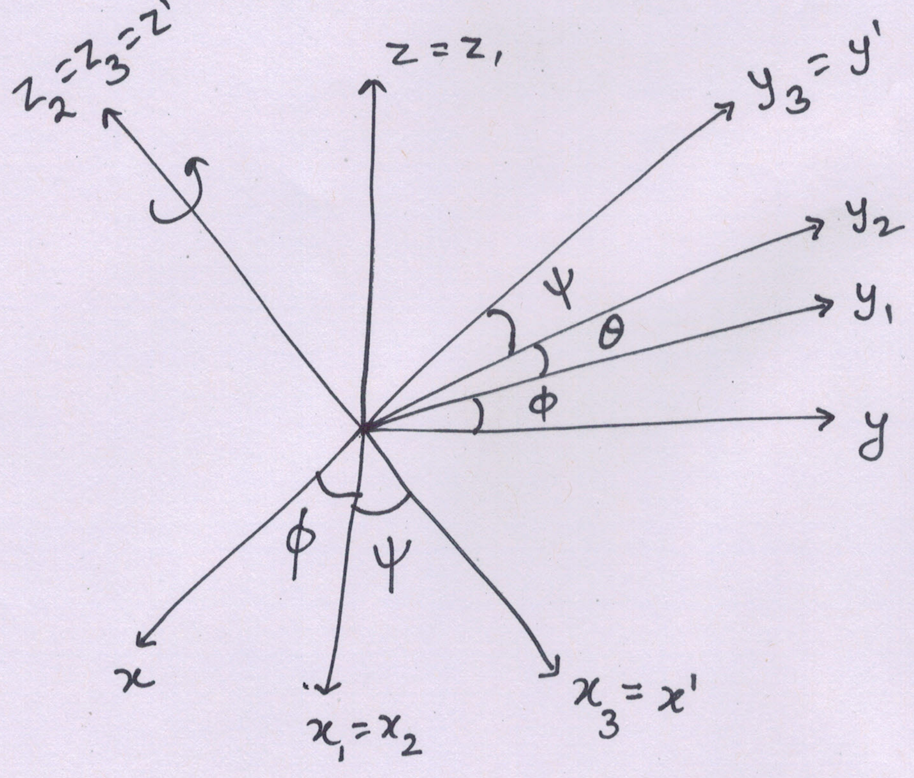
The transformation relation is,

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = C \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

Third rotation

In this rotation, co-ordinate (x_2, y_2, z_2) are rotated by an angle ψ in anticlockwise direction about the axis z_2. Then resulting co-ordinate axes are $(x_3, y_3, z_3) = (x', y', z')$

i.e. it coincide with body set of axes.



The transformation equations are;

$$\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = B \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

Motion of a symmetric top under action of gravity

Let us consider a symmetric top with one point fixed at origin. Let us take the distance of O from G (center of mass) to be (l).

Taking body z'-axis, the symmetry axis as the principal axis.

Let us put $I_1 = I_2$

Only force acting on the top is the force of gravity (mg) acting at G downwards.

The Lagrangian for the top can be written as;

$$L = T - V$$

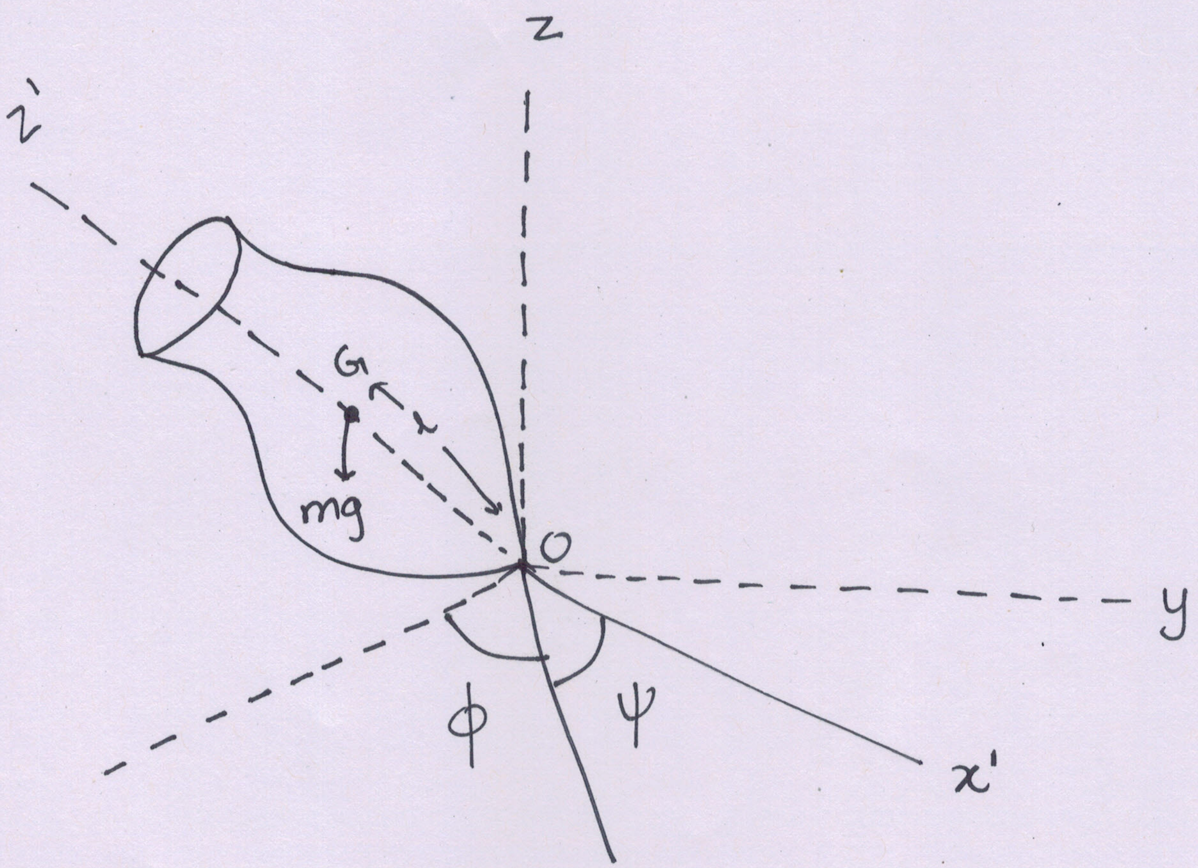
$$L = \frac{1}{2} I_1 \left(\omega_{x'}^2 + \omega_{y'}^2 \right) + \frac{1}{2} I_3 \omega_{z'}^2 - mgl\cos\theta$$

where,

$$\omega_{x'} = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$

$$\omega_{y'} = \dot{\phi} \sin\theta \cos\psi + \dot{\theta} \sin\psi$$

$$\omega_{z'} = \dot{\phi} \cos\theta + \dot{\psi}$$



On the substituting the values, the Lagrangian becomes;

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) + \frac{I_3}{2} \left(\dot{\psi} + \dot{\phi} \cos\theta \right)^2 - mgl \cos\theta \quad \text{—} (1)$$

Since the Lagrangian does not contain the Euler's angles ψ, ϕ and t i.e. they are cyclic.

The corresponding momenta P_ψ, P_ϕ and total energy(E) are constant in time.

i.e. $P_\psi = \dfrac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = I_3 \omega_{z'} = \cancel{I_1 a}$

$= I_1 a$

where (a) is constant. —(2)

&

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1 \dot{\phi} \sin^2\theta + I_3 \cos\theta(\dot{\psi} + \dot{\phi}\cos\theta)$$

$$= I_1 b \quad \text{—(3)}$$

where b is also constant.

and total energy $= T + V$

$$= \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}\omega_{z'}^2 + mgl\cos\theta$$

$$= E \quad \text{—(4)}$$