The corresponding momenta P_ψ, P_ϕ and total energy (E) are constant in time.

i.e. $$P_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = I_3 \omega_{z'} = \cancel{I_1 a}$$
$$= I_1 a$$

where (a) is constant. — (2)

&
$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1 \dot{\phi} \sin^2\theta + I_3 \cos\theta(\dot{\psi} + \dot{\phi}\cos\theta)$$
$$= I_1 b \quad — (3)$$

where b is also constant.

and total energy = T + V

$$= \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}\omega_{z'}^2 + mgl\cos\theta$$

$$= E \quad — (4)$$

From eq^n (3),

$$I_1 \dot{\phi} \sin^2\theta + I_3 \cos^2\theta\, \dot{\phi} + I_3 \dot{\psi}\cos\theta = I_1 b$$

Using value of $I_3 \dot{\psi}$ from eq^n (2) in above eq^n:

$$I_3 \dot{\psi} = I_1 a - I_3 \dot{\phi} \cos\theta \quad — (4a)$$

$$I_1 \dot{\phi} \sin^2\theta + I_3 \cos^2\theta \, \dot{\phi} + (I_1 a - I_3 \dot{\phi} \cos\theta)\cos\theta = I_1 b$$

or, $$I_1 \dot{\phi} \sin^2\theta + \cancel{I_3 \cos^2\theta \, \dot{\phi}} + I_1 a \cos\theta - \cancel{I_3 \dot{\phi} \cos^2\theta} = I_1 b$$

or, $$\dot{\phi}\left(I_1 \sin^2\theta\right) = I_1 (b - a\cos\theta)$$

or, $$\dot{\phi} = \frac{\cancel{I_1}(b - a\cos\theta)}{\cancel{I_1} \sin^2\theta}$$

or, $$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \quad \text{— (5)}$$

Putting value of $\dot{\phi}$ in eq^n (4a).

$$I_3 \dot{\psi} = I_1 a - I_3 \left(\frac{b - a\cos\theta}{\sin^2\theta}\right)\cos\theta$$

$$\dot{\psi} = \frac{I_1}{I_3} a - \left(\frac{b - a\cos\theta}{\sin^2\theta}\right)\cos\theta \quad \text{— (6)}$$

Total energy of the top is given by,

$$E = T + V$$

$$E = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + mgl\cos\theta$$

or, $E - \frac{I_3}{2}(\dot\psi + \dot\phi\cos\theta)^2 = \frac{1}{2} I_1 \left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right)$

$+ mgl\cos\theta$

or, $$E' = \frac{I_1}{2}\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + mgl\cos\theta \quad \text{—⑦}$$

where, $E' = E - \frac{I_3}{2}(\dot\psi + \dot\phi\cos\theta)^2$.

Substituting the value of $\dot\phi$, in eqn ⑦,

$$E' = \frac{1}{2} I_1 \dot\theta^2 + \frac{1}{2} I_1 \sin^2\theta \left(\frac{b - a\cos\theta}{\sin^2\theta}\right)^2 + mgl\cos\theta$$

$$E' = \frac{1}{2} I_1 \dot\theta^2 + \frac{1}{2} I_1 \frac{(b - a\cos\theta)^2}{\sin^2\theta} + mgl\cos\theta$$

Multiplying both sides by $\frac{2\sin^2\theta}{I_1}$,

$$\frac{2E'\sin^2\theta}{I_1} = \sin^2\theta\,\dot\theta^2 + (b - a\cos\theta)^2 + \frac{2mgl}{I_1}\sin^2\theta\cos\theta$$

Let $\frac{2E'}{I_1} = \alpha$ & $\beta = \frac{2mgl}{I_1}$

or, $$\alpha\sin^2\theta = \sin^2\theta\,\dot\theta^2 + (b - a\cos\theta)^2 + \beta\sin^2\theta\cos\theta$$

Let, $u = \cos\theta$; $\dot u = -\sin\theta\,\dot\theta$

or, $\alpha(1-u^2) = \dot{u}^2 + (b-au)^2 + \beta(1-u^2)u$

or, $\dot{u}^2 = (\alpha-\beta u)(1-u^2) - (b-au)^2$

or, $\dot{u} = \sqrt{[(\alpha-\beta u)(1-u^2) - (b-au)^2]}$

or, $\frac{du}{dt} = \sqrt{[(\alpha-\beta u)(1-u^2) - (b-au)^2]}$

or, $dt = \frac{du}{\sqrt{(\alpha-\beta u)(1-u^2)-(b-au)^2}}$

or, $\int_0^t dt = t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(\alpha-\beta u)(1-u^2) - (b-au)^2}}$

where, $u = u(0)$ at $t=0$ & $u = u(t)$ at $t=t$.

This expression gives relation between θ & t. This is the integral equation of motion of symmetrical top. Knowing the relation $\theta = \theta(t)$, the equation $\psi = \psi(t)$ and $\phi = \phi(t)$ can be easily solved.

Angular velocity in terms of Euler's Angles

We have,

$$A = BCD$$

$$A = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} (\omega_\phi)_{x'} \\ (\omega_\phi)_{y'} \\ (\omega_\phi)_{z'} \end{bmatrix} = A \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$\left. \begin{aligned} (\omega_\phi)_{x'} &= \dot{\phi} \sin\theta \sin\psi \\ (\omega_\phi)_{y'} &= \dot{\phi} \sin\theta \cos\psi \\ (\omega_\phi)_{z'} &= \dot{\phi} \cos\theta \end{aligned} \right\} \quad - (1)$$

Again,

$$\begin{bmatrix} (\omega_\theta)_{x'} \\ (\omega_\theta)_{y'} \\ (\omega_\theta)_{z'} \end{bmatrix} = B \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}$$

$$\left. \begin{aligned} (\omega_\theta)_{x'} &= \dot{\theta} \cos\psi \\ (\omega_\theta)_{y'} &= -\dot{\theta} \sin\psi \\ (\omega_\theta)_{z'} &= 0 \end{aligned} \right\} \quad - (2)$$

Finally,

$$\begin{bmatrix} (\omega_\psi)_{x'} \\ (\omega_\psi)_{y'} \\ (\omega_\psi)_{z'} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} (\omega_\psi)_{x'} &= 0 \\ (\omega_\psi)_{y'} &= 0 \\ (\omega_\psi)_{z'} &= \dot{\psi} \end{aligned}\right\} \quad (3)$$

Now,

$$\omega_{x'} = (\omega_\phi)_{x'} + (\omega_\theta)_{x'} + (\omega_\psi)_{x'}$$

$$\omega_{x'} = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$

$$\omega_{y'} = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$$

$$\omega_{z'} = \dot{\phi} \cos\theta + \dot{\psi}$$