卐 Chapter-7: Motion in Central Field

Syllabus: 7.1. Motion in central force field, motion in arbitrary potential field, equation of orbits, 7.2 Kepler's Laws of planetary Motion

Introduction (Central Force)

The force which is always directed towards or away from a fixed center and the magnitude of which is a function of the distance from the center to center of mass of the particle is called central force.

i.e. $F(r) = f(r)\,\hat{r}$

where, $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$ is a unit vector along the radius vector r from the fixed center.

$f(r)$ is the function of distance only.

For attractive force, $f(r) < 0$

For Repulsive force, $f(r) > 0$

Force between two masses m_1 & m_2; $f = \dfrac{GMm}{r^2}$

$\because f \propto \dfrac{1}{r^2}$, so gravitational force is also central.

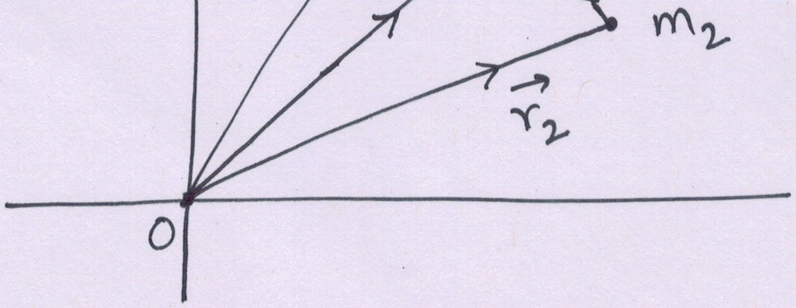
卐 Properties of Central Force:

1. Central forces are long range forces.
2. The magnitude of the central force depends only on the distance between the centers of the two bodies.
3. It acts along the line joining the centers of two bodies.
4. As the workdone during (under) central force is independent of the path, it is conservative force.

Long: Show that two body problem can be reduced into single body problem

OR

Equivalent One body problem:



Let us consider a conservative system of two mass points m_1 and m_2. having position vector $\vec{r_1}$ and $\vec{r_2}$. Let CM represents center of mass of the system having position vector $\vec{R}$.

The Lagrangian for such a system can be written as,

$$L = T(\dot{R}, \dot{r}) - V(r) \quad \text{—} \ (1)$$

where,

$$T(\dot{R}, \dot{r}) = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + T'$$

which expresses the KE as the sum of KE of motion of the center of mass plus the KE of motion about the center of mass T'. It has been assumed that V is purely a function of (r).

Here,

$$T' = \frac{1}{2} m_1 \dot{r}_1'^2 + \frac{1}{2} m_2 \dot{r}_2'^2$$

where, $\vec{r'}$ and $\vec{r_2'}$ are the position vectors with respect to center of mass.

Here, $$\vec{r} = \vec{r_1} - \vec{r_2} = \vec{r_1'} - \vec{r_2'} \quad \text{—} \ (2)$$

As the body balances about the C.M.

Thus, $m_1 \overrightarrow{r_1'} + m_2 \overrightarrow{r_2'} = 0$

$$r_1' = -\left(\frac{m_2}{m_1}\right) r_2'$$

Neglecting vector sign.

Putting value of r_2' in eqn ②.

$$r = r_1' + \frac{m_1}{m_2} r_1'$$

$$r = \left(\frac{m_1 + m_2}{m_2}\right) r_1'$$

or, $$r_1' = \left(\frac{m_2}{m_1+m_2}\right) r$$

& $$r_2' = -\left(\frac{m_1}{m_1+m_2}\right) r$$

Now, $$KE = \frac{1}{2}(m_1+m_2)\dot{R}^2 + \frac{1}{2} m_1 \dot{r}_1'^2 + \frac{1}{2} m_2 \dot{r}_2'^2$$

$$KE = \frac{1}{2}(m_1+m_2)\dot{R}^2 + \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 \dot{r}^2 + \frac{1}{2} m_2 \left(\frac{m_1}{m_1+m_2}\right)^2 \dot{r}^2$$

$$T = \frac{1}{2}(m_1+m_2)\dot{R}^2 + \frac{1}{2}\frac{m_1 m_2}{(m_1+m_2)}\dot{r}^2$$

$$T = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu \dot{r}^2$$

where, $M = m_1 + m_2$

$$\mu = \frac{m_1 m_2}{m_1+m_2} \text{ (reduced mass)}$$

Thus, Lagrangian

$$(L) = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu \dot{r}^2 - V(r)$$

The Lagrange's equation of motion for two variables R & r can be written as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) - \frac{\partial L}{\partial R} = 0$$

or, $$\frac{d}{dt}(M\dot{R}) = 0$$

Or, $$M\dot{R} = \text{constant}$$

or, $$\dot{R} = \text{constant}$$

This implies velocity of center of mass is constant.

Again for variable ~~w~~. r,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

or, $$\frac{d}{dt}\left(\mu \dot{r}\right) + \frac{\partial V(r)}{\partial r} = 0$$

or, $$\mu \ddot{r} = -\frac{\partial V(r)}{\partial r}$$

or, $$\mu \ddot{r} = f(r)$$

This eqn does not involve R or $\dot{R}$. so $\frac{1}{2} M \dot{R}^2$ can be omitted.

which represents equation of motion for the system under consideration.

The Lagrangian $\boxed{(L) = \frac{1}{2} \mu \dot{r}^2 - V(r)}$ —③

By neglecting term $\frac{1}{2} M \dot{R}^2$.

This eqn ③ is equivalent to Lagrangian of particle of mass is at a distance (r) from the fixed center which provides potential energy $V(r)$. In this way two body problem in central force field can be reduced to equivalent single body problem.