卐 Kepler's Laws of Planetary Motion

First Law: Each planet revolves around the sun in an elliptical orbit with the sun at one of the foci. This law gives shape of the orbit of planets around the sun.

Second law: The radius vector joining sun to planet sweeps out the equal area in equal interval of time. i.e. the areal velocity is constant.

Third law: The square of time period of revolution of planet round the sun is directly proportional to cube of semi-major axis. i.e. $T^2 \propto a^3$

Proof of Kepler's First Law

Let us consider a body of mass (m) moving in a central force field, where potential is given by,

$$V(r) = \frac{-k}{r}$$

The total energy of the body, $E = KE + PE$

In polar co-ordinates, $KE = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$

But angular momentum (l) is given by,

$$l = m r^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{l}{m r^2}$$

So,

$$KE = \frac{1}{2} m \left(\dot{r}^2 + r^2 \frac{l^2}{m^2 r^4} \right)$$

$$KE = \frac{1}{2} m \left(\dot{r}^2 + \frac{l^2}{m^2 r^2} \right)$$

Hence total energy $(E) = \frac{1}{2} m \left(\dot{r}^2 + \frac{l^2}{m^2 r^2} \right) - \frac{k}{r}$ —— (1)

In a conservative field, total energy remains constant. Consider the body is at turning point at which $r = r_{min}$.

Thus, at turning point $r = r_{min}$ = constant

so $\dot{r} = 0$

So eq^n ① becomes,

$$E = \frac{l^2}{2m r_{min}^2} - \frac{k}{r_{min}} \quad \text{—— ②}$$

The equation of conic section is,

$$r = \frac{p}{1 + \epsilon \cos\theta}$$

where, p - semi latus rectum

ϵ - eccentricity

θ - Angle between radius vector & semi major axis.

For $r = r_{min}$, $\theta = 0$ so $\cos\theta = 1$

$$\therefore \quad r_{min} = \frac{p}{1 + \epsilon}$$

We know, $p = \dfrac{l^2}{mK}$

$$\therefore \quad r_{min} = \frac{l^2}{mK(1+\epsilon)}$$

Then eq^n (2) becomes,

or, $$E = \frac{l^2}{2m} \cdot \frac{m^2k^2(1+\epsilon)^2}{l^4} - \frac{k \cdot mk(1+\epsilon)}{l^2}$$

or, $$E = \frac{mk^2}{2l^2}\left[1+2\epsilon+\epsilon^2-2-2\epsilon\right]$$

or, $$E = \frac{mk^2}{2l^2}(\epsilon^2-1)$$

or, $$\frac{2l^2E}{mk^2} = \epsilon^2-1$$

or, $$\epsilon = \sqrt{1+\frac{2El^2}{mk^2}}$$

which is the eccentricity of the path of the body.

We also have,

$$r = \frac{p}{1+\epsilon\cos\theta}$$

or, $$\frac{p}{r} = 1+\epsilon\cos\theta$$

or, $$\frac{\frac{l^2}{mk}}{r} = 1+\sqrt{1+\frac{2El^2}{mk^2}}\cos\theta$$

or, $$\frac{l^2}{mkr} = 1 + \sqrt{1 + \frac{2El^2}{mk}}\cos\theta$$

As the angular momentum $l = mr^2\dot{\theta}$ is a constant quantity in a central force field, the nature of path of the body depends upon it's total energy.

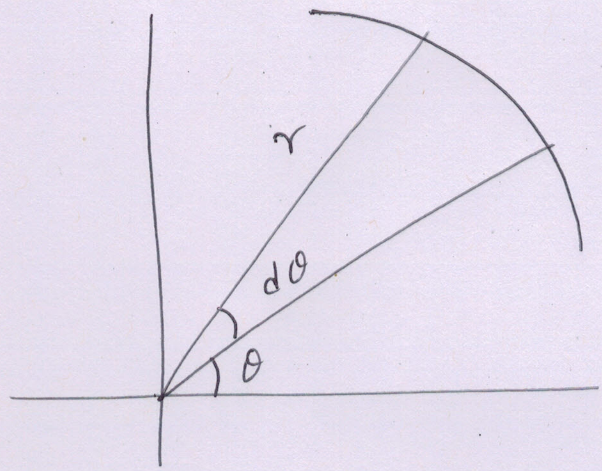
Cases:

1. If $E > 0$; $\epsilon > 1$ then the path will be hyperbolic
2. If $E = 0$; $\epsilon = 1$ then the path will be parabolic.
3. If $-\frac{mk^2}{2l^2} < E < 0$; $\epsilon < 1$ then the path will be elliptical.
4. If $E = \frac{-mk^2}{2l^2}$; $\epsilon = 0$ then the path will be circular

For planetary motion, $E < 0$ hence each planet revolves around the sun in an elliptical orbit with sun at one of the foci.

This proves Kepler's First law.

Proof of Kepler's 2^{nd} Law



The Lagrangian of the particle of mass (m) moving in the central force field is given by,

$$L = T - V$$

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \quad \text{——} \ (1)$$

In this equation generalized co-ordinate (θ) is cyclic so generalized momentum conjugate to θ is constant.

i.e. $$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = \text{constant} = l \text{ (say)}$$

or, $$l = \mu r^2 \dot{\theta}$$

or, $$\frac{l}{2\mu} = \frac{r^2\dot{\theta}}{2}$$

Suppose radius vector r makes an angle $(d\theta)$ in a small interval of time dt.

Then area swept out by radius vector is,

$$dA = \frac{1}{2}\cdot r \cdot r\,d\theta$$

or, $dA = \frac{1}{2} r^2 d\theta$

or, $\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$

or, $\frac{dA}{dt} = \frac{l}{2\mu}$

Since (l) and (μ) are constant, $\frac{dA}{dt} =$ constant

which means areal velocity of planet is constant.

Kepler's Third Law $(T^2 \propto a^3)$

Let us consider a planet moving around the sun in an elliptical orbit of semi-major axis (a) and semi-minor axis (b) and latus rectum p.

Then area of the elliptical orbit is $(A) = \pi ab$

$$\text{Time period } (T) = \frac{\text{Area of ellipse}}{\text{Areal velocity}}$$

$$(T) = \frac{\pi ab}{(l/2\mu)}$$

$\because$ Since areal velocity $\frac{dA}{dt} = \frac{l}{2\mu}$

So,

$$T = \frac{\pi ab}{\left(\frac{l}{2\mu}\right)}$$

or,

$$T = \frac{2\pi \mu ab}{l}$$

Squaring on both sides,

$$T^2 = \frac{4\pi^2 \mu^2 a^2 b^2}{l^2}$$

Again latus rectum $(p) = \frac{b^2}{a} \Rightarrow b^2 = pa$

or,

$$T^2 = \frac{4\pi^2 \mu^2 a^2 . pa}{l^2}$$

$$T^2 = \left(\frac{4\pi^2 \mu^2 p}{l^2}\right) a^3$$

If mass(m) of the planet is very small compared to mass(M) of the sun. Then, $\mu = \frac{mM}{m+M} \approx m$

$$T^2 = \left(\frac{4\pi^2 m^2 p}{l^2}\right) a^3$$

$$T^2 = K a^3 \Rightarrow \boxed{T^2 \propto a^3}$$

<u>proved</u>

Q. A particle is moving describing the path $r = \frac{1}{1+\epsilon\cos\theta}$; under the action of central force. Shows that force varies as square of radius vector inversly.

Hint

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2 u^2} f\left(\frac{1}{u}\right)$$

$$u = \frac{1}{r} = 1 + \epsilon\cos\theta.$$

$$\frac{du}{d\theta} = -\epsilon\sin\theta$$

$$\frac{d^2u}{d\theta^2} = -\epsilon\cos\theta$$

$$\Rightarrow f\left(\frac{1}{u}\right) = \frac{-l^2}{\mu} u^2$$

$$f(r) = \frac{-l^2}{\mu r^2}$$

$$f(r) = -\left(\frac{l^2}{\mu}\right)\frac{1}{r^2}$$

$$\boxed{f(r) \propto \frac{1}{r^2}}$$

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