Classical Mechanics: B.Sc. Third year: Chapter -Elementary Principles

D'Alembert's Principle

Let us suppose a system is in equilibrium i.e. the total force $\boldsymbol{F_i}$ on every particle is zero then workdone by this force in a small virtual displacement $\delta \boldsymbol{r_i}$ will also vanish. i.e for whole system for N particles

$$\sum_{i=1}^{N} \boldsymbol{F_i} \,.\, \delta \boldsymbol{r_i} = 0 \ldots \ldots \ldots \ldots (1)$$

Let this total force be expressed as sum of applied force $\boldsymbol{F_i^a}$ and forces of constraints $\boldsymbol{f_i}$. Then above equation takes the form,

$$\sum_{i} \boldsymbol{F_i^a} \,.\, \delta \boldsymbol{r_i} + \sum_{i} \boldsymbol{f_i} \,.\, \delta \boldsymbol{r_i} = 0 \ldots \ldots \ldots (2)$$

We now consider the system for which the virtual work of the forces of constraint is zero. Thus,

$$\sum_{i} \boldsymbol{F_i^a} \,.\, \delta \boldsymbol{r_i} = 0 \ldots \ldots \ldots \ldots \ldots (3)$$

The equation is termed as principle of virtual work.

To interpret the equilibrium of the system, D'Alembert adopted an idea of a reversed force.

He considered that a system will remain in equilibrium under the action of a force equal to the actual force $\boldsymbol{F_i}$ plus reversed effective force $\dot{\boldsymbol{p}}_i$.

Thus,

$$\boldsymbol{F_i} + (-\dot{\boldsymbol{p}}_i) = 0$$

$$\boldsymbol{F_i} - \dot{\boldsymbol{p}}_i = 0$$

where, $\dot{\boldsymbol{p}}_i$ represents an effective force called reversed force of inertia on the i^{th} particle.

Thus principle of virtual work takes the form,

$$\sum_{i} (\boldsymbol{F_i} - \dot{\boldsymbol{p}}_i).\, \delta \boldsymbol{r_i} = 0 \ldots \ldots (4)$$

Again putting , $\boldsymbol{F_i} = \boldsymbol{F_i^a} + \boldsymbol{f_i}$ in eq(4);

$$\sum_i (\boldsymbol{F}_i^a - \dot{\boldsymbol{p}}_i).\delta \boldsymbol{r}_i + \sum_i \boldsymbol{f}_i.\delta \boldsymbol{r}_i = 0$$

Since there is no constraint, $\boldsymbol{f}_i = 0$. Then

$$\sum_i (\boldsymbol{F}_i^a - \dot{\boldsymbol{p}}_i).\delta \boldsymbol{r}_i = 0$$

To write in generalised form omit superscript a. Hence we get

$$\sum_i (\boldsymbol{F}_i - \dot{\boldsymbol{p}}_i).\delta \boldsymbol{r}_i = 0$$

Which is called D'Alembert's Principle.

#Derivation of Lagrange's equation from D'Alembert's Principle

The co-ordinate transformation equations are ;

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, q_3 \dots \dots q_n, t)$$

So that;

$$\frac{d\boldsymbol{r}_i}{dt} = \frac{\partial \boldsymbol{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \boldsymbol{r}_i}{\partial q_2}\frac{dq_2}{dt} + \cdots \dots + \frac{\partial \boldsymbol{r}_i}{\partial t}\frac{dt}{dt}$$

$$\boldsymbol{v}_i = \sum_j \frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t} \dots\dots (1)$$

Further infinitesimal displacement δr_i can be written as;

$$\delta \boldsymbol{r}_i = \sum_j \frac{\partial \boldsymbol{r}_i}{\partial q_j}\delta q_j + \frac{\partial \boldsymbol{r}_i}{\partial t}\delta t$$

Here, last term is zero since in virtual displacement only coordinate displacement is considered and not that of time. Therefore;

$$\delta \boldsymbol{r}_i = \sum_j \frac{\partial \boldsymbol{r}_i}{\partial q_j}\delta q_j$$

From D'Alembert's Principle,

$$\sum_i (\boldsymbol{F_i} - \dot{\boldsymbol{p}}_i).\delta \boldsymbol{r_i} = 0$$

Putting value of $\delta \boldsymbol{r_i}$

$$\sum_i (\boldsymbol{F_i} - \dot{\boldsymbol{p}}_i).\ \sum_j \frac{\partial \boldsymbol{r_i}}{\partial q_j}\ \delta q_j\ = 0$$

$$\sum_{ij} \boldsymbol{F_i}.\frac{\partial \boldsymbol{r}_i}{\partial q_j}\ \delta q_j - \sum_{ij} \dot{\boldsymbol{p}}_i.\ \frac{\partial \boldsymbol{r_i}}{\partial q_j}\ \delta q_j = 0$$

Let us define ;

$$\sum_i \boldsymbol{F_i} .\frac{\partial \boldsymbol{r_i}}{\partial q_j} = Q_j\ \ ; called\ generalised\ force.$$

Then above equation takes the form;

$$\sum_j Q_j\ \delta q_j - \sum_{ij} \dot{\boldsymbol{p}}_i.\ \frac{\partial \boldsymbol{r_i}}{\partial q_j}\ \delta q_j = 0 \ldots\ldots (2)$$

Let us simplify the second term;

$$\sum_{ij} \dot{\boldsymbol{p}}_i.\ \frac{\partial \boldsymbol{r_i}}{\partial q_j}\ \delta q_j = \sum_{ij} m_i\ \ddot{\boldsymbol{r}}_i\ .\ \frac{\partial \boldsymbol{r_i}}{\partial q_j}\ \delta q_j$$

$$= \sum_{ij} \left\{\frac{d}{dt}\left(m_i\ \dot{\boldsymbol{r}}_i\ .\frac{\partial \boldsymbol{r_i}}{\partial q_j}\right) - m_i\ \dot{\boldsymbol{r}}_i.\frac{d}{dt}\left(\frac{\partial \boldsymbol{r_i}}{\partial q_j}\right)\right\}\ \delta q_j$$

$$\sum_{ij} \dot{\boldsymbol{p}}_i.\ \frac{\partial \boldsymbol{r}_i}{\partial q_j}\ \delta q_j\ = \sum_{ij} \left\{\frac{d}{dt}\left(m_i\ \boldsymbol{v_i}\ .\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) - m_i\ \boldsymbol{v_i}.\frac{d}{dt}\left(\frac{\partial \boldsymbol{r_i}}{\partial q_j}\right)\right\}\ \delta q_j \ldots . (3)$$

$$\sum_{ij} \dot{\boldsymbol{p}}_i.\ \frac{\partial \boldsymbol{r_i}}{\partial q_j}\ \delta q_j\ = \sum_{ij} \left\{\frac{d}{dt}\left(m_i\ \boldsymbol{v_i}\ .\frac{\partial \boldsymbol{r_i}}{\partial q_j}\right) - m_i\ \boldsymbol{v_i}.\frac{\partial}{\partial q_j}\left(\frac{d\boldsymbol{r_i}}{dt}\right)\right\}\ \delta q_j$$

$$\sum_{ij} \dot{\boldsymbol{p}}_i.\ \frac{\partial \boldsymbol{r_i}}{\partial q_j}\ \delta q_j\ = \sum_{ij} \left\{\frac{d}{dt}\left(m_i\ \boldsymbol{v_i}\ .\frac{\partial \boldsymbol{r_i}}{\partial q_j}\right) - m_i\ \boldsymbol{v_i}.\frac{\partial \boldsymbol{v_i}}{\partial q_j}\right\}\ \delta q_j$$

Again; $\frac{\partial \boldsymbol{r_i}}{\partial q_j} = \frac{\partial \boldsymbol{v_i}}{\partial \dot{q}}$

$$\sum_{ij} \dot{\boldsymbol{p}}_{\boldsymbol{\iota}}.\ \frac{\partial \boldsymbol{r_i}}{\partial q_j}\ \delta q_j = \sum_{ij} \left\{ \frac{d}{dt}\left(m_i\ \boldsymbol{v_i}\ .\frac{\partial \boldsymbol{v_i}}{\partial \dot{q}} \right) - m_i\ \boldsymbol{v_i}.\frac{\partial \boldsymbol{v_i}}{\partial q_j} \right\}\ \delta q_j$$

$$\sum_{ij} \dot{\boldsymbol{p}}_{\boldsymbol{\iota}}.\ \frac{\partial \boldsymbol{r_i}}{\partial q_j}\ \delta q_j = \sum_{j} \left[\frac{d}{dt}\left\{ \frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right\} - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right]\ \delta q_j$$

$$\sum_{ij} \dot{p}_{\iota}.\ \frac{\partial r_i}{\partial q_j}\ \delta q_j = \sum_{j} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j$$

Putting this value in eq (2) then

$$\sum_j Q_j\ \delta q_j - \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = 0$$

$$\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right]\ \delta q_j = 0$$

By making the coefficients of δq_j to be zero, we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \ \ldots \ldots (4)$$

Case I: For conservative system

In conservative system potential energy is the function of coordinate only. i.e. $V = V(r)$

$$\boldsymbol{F_i} = -\frac{\partial V}{\partial \boldsymbol{r_i}}$$

Then generalized force can be written as;

$$Q_j = \sum_i \boldsymbol{F_i}\ .\frac{\partial r_i}{\partial q_j} = \sum_i -\frac{\partial V}{\partial \boldsymbol{r_i}}\ .\frac{\partial \boldsymbol{r_i}}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

Putting value of Q_j in eq (4), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial (T - V)}{\partial \dot{q}_j}\right) - \frac{\partial (T - V)}{\partial q_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Which is required equation.

where $L = T - V; represents\ Lagrangian.$

Above equation is called Lagrange's equation of motion for conservative system.

Case II: Non-conservative system

Here potential energy is velocity dependent so geeralised force can be written as;

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

Now from eq(4);

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

$$\frac{d}{dt}\left(\frac{\partial (T - U)}{\partial \dot{q}_j}\right) - \frac{\partial (T - U)}{\partial q_j} = 0$$

If we put Lagrangian for non-conservative system as $L = T - U$ then above equation becomes;

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$