B.Sc. Third year (Classical Mechanics)

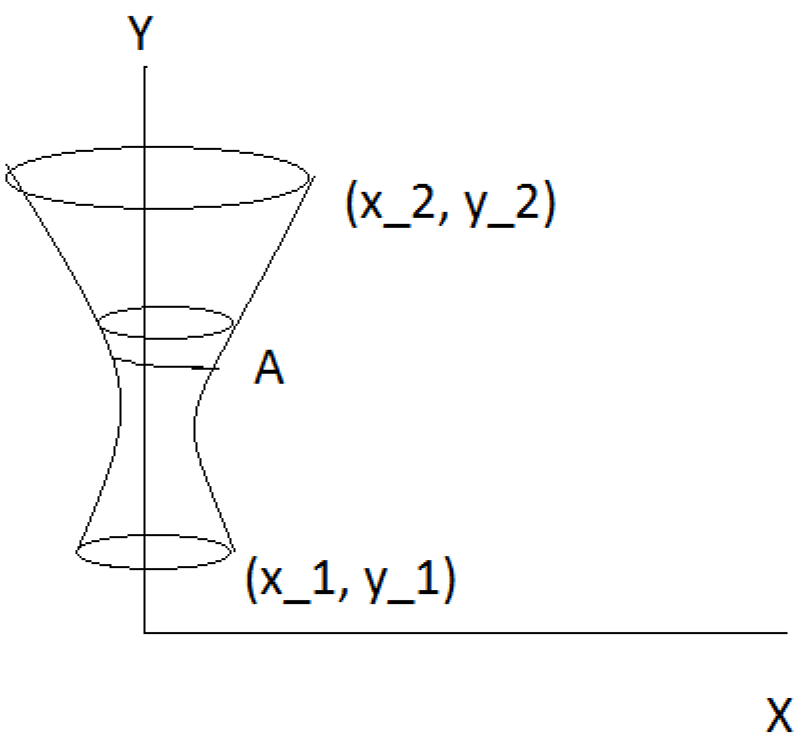
Chapter 10- Variational Principles and Lagrange's equations

Lecture 2

Examples of Calculus of Variation

Minimum surface of revolution

We form a surface of revolution by revolving a curve about a certain axis. In this example, a curve passing through two ends (x_1, y_1) and (x_2, y_2) has been rotated about y-axis.



Let us consider a strip at point A formed due to the revolution of the arc length ds about Y-axis. If the distance of this arc from y-axis is x then the surface area of the strip $2\pi x\ ds$

$$= 2\pi x \sqrt{1 + (\dot{y})^2}\, dx$$

The total surface area is then

$$I = \int_1^2 2\pi x\sqrt{1 + (\dot{y})^2}\, dx = \int_1^2 f\, dx$$

And it will be minimum if $\delta I = 0$ for which the equation,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \ldots\ldots (i)$$

Here $f = 2\pi x\sqrt{1 + (\dot{y})^2}$ so that

$$\frac{\partial f}{\partial y} = 0; \quad \frac{\partial f}{\partial \dot{y}} = 2\pi \frac{x\dot{y}}{\sqrt{1 + (\dot{y})^2}}$$

Putting these values in eq (i),

$$\frac{d}{dx}\left(2\pi \frac{x\dot{y}}{\sqrt{1 + (\dot{y})^2}}\right) = 0$$

$$\frac{x\dot{y}}{\sqrt{1 + (\dot{y})^2}} = a\ (say)$$

Squaring,

$$x^2(\dot{y})^2 = a^2 + a^2(\dot{y})^2$$

$$(\dot{y})^2 = \frac{a^2}{x^2 - a^2}$$

$$\dot{y} = \frac{a}{\sqrt{x^2 - a^2}}$$

On integrating,

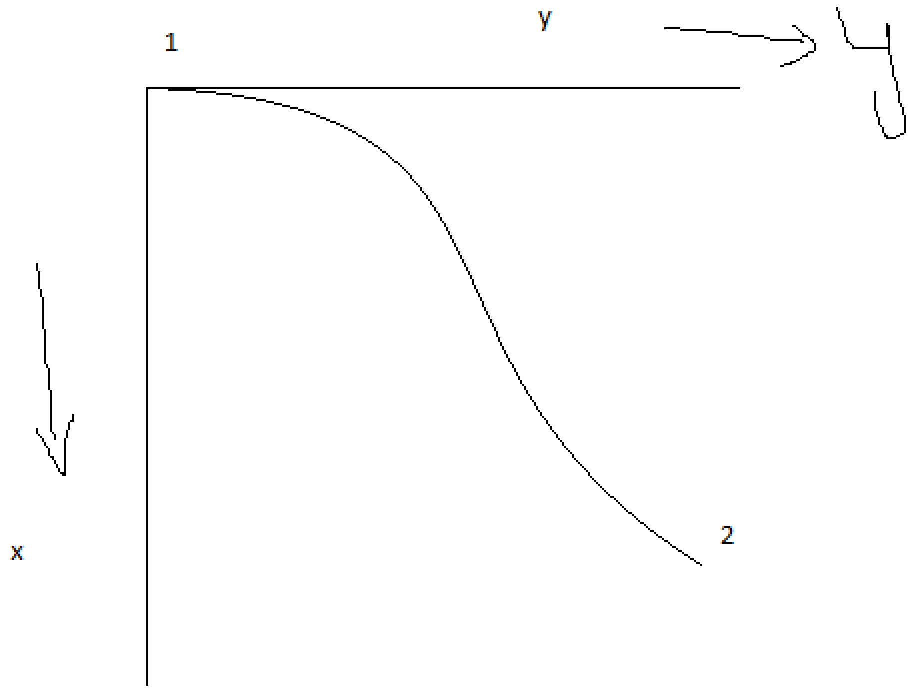
$$y = a\cosh^{-1}\frac{x}{a} + b$$

$$x = a\cosh\frac{y - b}{a}$$

Which is the equation of catenary.

The Branchistochrone problem

In this problem we find a curve joining two points along which a particle falling from rest under the influence of gravity travels from higher to lower point in the minimum time.



Suppose v is the speed of particle along the curve then in traversing ds portion of the curve time spent will be $\frac{ds}{v}$ so that the total time taken by particle in moving from heighest point 1 to lowest point 2 is ,

$$t_{12} = \int_1^2 \frac{ds}{v}$$

Suppose vertical distance of fall upto point 2 be x then from principle of conservation of energy,

$$\frac{1}{2} mv^2 = mgx$$

$$v = \sqrt{2\ gx}$$

$$t_{12} = \int_1^2 \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gx}} = \int_1^2 \frac{\sqrt{1 + (\dot{y})^2}}{\sqrt{2gx}}\ dx = \int_1^2 f\ dx$$

Where, $f = \frac{\sqrt{1+(\dot{y})^2}}{\sqrt{2gx}}$

For t_{12} to be minimum it must satisfy the equation,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \ldots\ldots (i)$$

$$\frac{\partial f}{\partial y} = 0; \quad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{2gx}\sqrt{1+(\dot{y})^2}}$$

Now putting values in eq (i)

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{2gx}\sqrt{1+(\dot{y})^2}}\right) = 0$$

$$\frac{\dot{y}}{\sqrt{x}\sqrt{1+(\dot{y})^2}} = constant$$

$$\frac{(\dot{y})^2}{x(1+(\dot{y})^2)} = c$$

$$\frac{(\dot{y})^2}{c} = x(1+(\dot{y})^2)$$

$$(\dot{y})^2\left(\frac{1}{c} - x\right) = x$$

$$(\dot{y})^2\left(\frac{x}{c} - x^2\right) = x^2$$

$$\dot{y} = \frac{x^2}{\left(\frac{x}{c} - x^2\right)^{\frac{1}{2}}}$$

Let us put, $\frac{1}{c} = 2a$ now after integration we get

$$y = a\cos^{-1}\left(1 - \frac{x}{a}\right) - (2ax - x^2)^{\frac{1}{2}}$$

Which represents an inverted cycloid with the base along y axis and cusp at the origin.