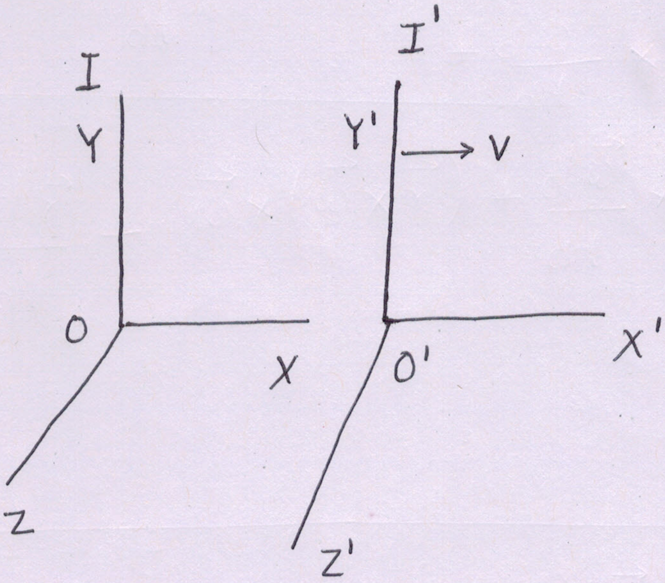
Chapter 11: Inertial Frames

Syllabus: Moving co-ordinate system, translating and rotating co-ordinate system ; Coriolis force ; Foucault pendulum

Translational Motion

Let us consider two inertial frames I and I' such that I' is moving along X-axis with velocity(v) and their axes coincide at t=0.



Then transformation relation relating position and time are:

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

These equations are said to be defined as a Galilean transformation.

The corresponding components of velocity are;

$$V'_x = V_x - v \quad , \quad V'_y = V_y \quad , \quad V'_z = V_z \, .$$

And acceleration components are;

$$a'_x = a_x \quad , \quad a'_y = a_y \quad , \quad a'_z = a_z$$

If the particle of mass(m) is acted upon by force (F) in the system(I) then

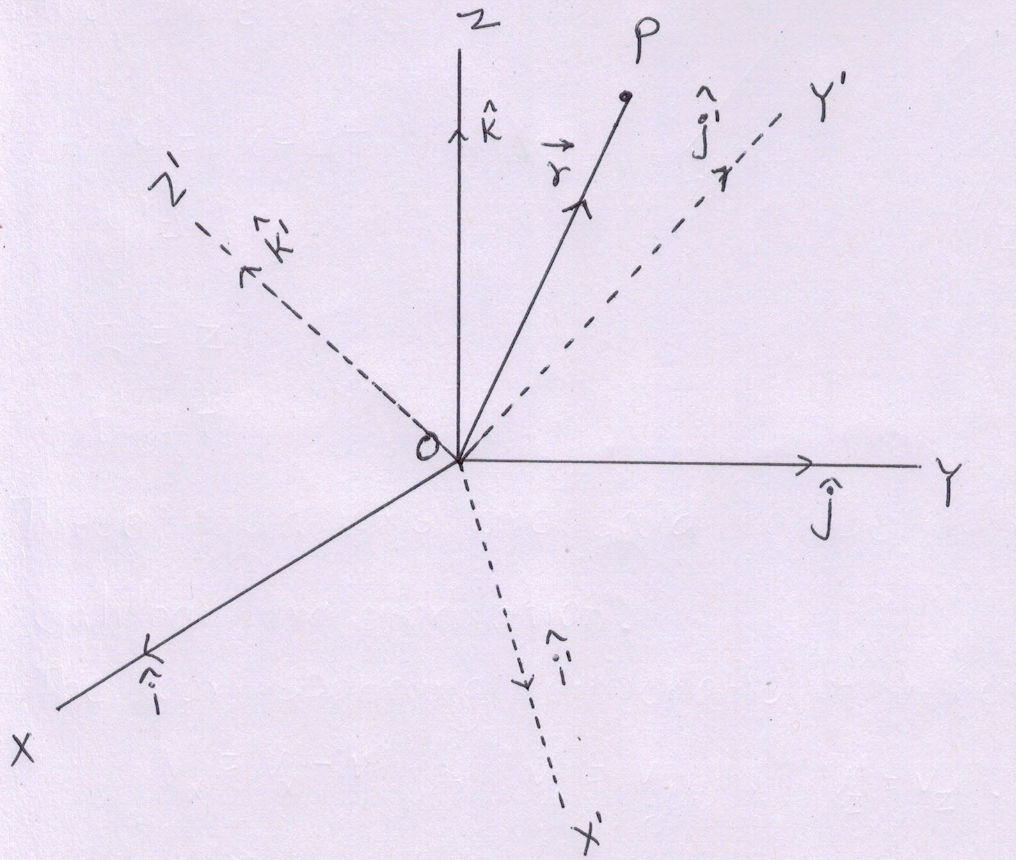
$$F = ma$$

A similar equation holds good in frame I' also;

$$F' = ma'$$

Hence in two inertial frames, the Newton's laws of motion remain in the same form i.e. they are invariant

Rotating co-ordinate System



Let us consider a called reference frame XYZ fixed in space and another reference frame X'Y'Z' fixed in the body.

Let at time $t=0$, their origins and base vectors Coincide.

When the body rotates, the frame OX'Y'Z' will rotate with respect to the frame OXYZ

Let us take any point $P(x,y,z)$ in the unprimed system (also called space set of axes or stationary co-ordinate system) is described by the position vector $\vec{r}$. The motion of P is therefore described by,

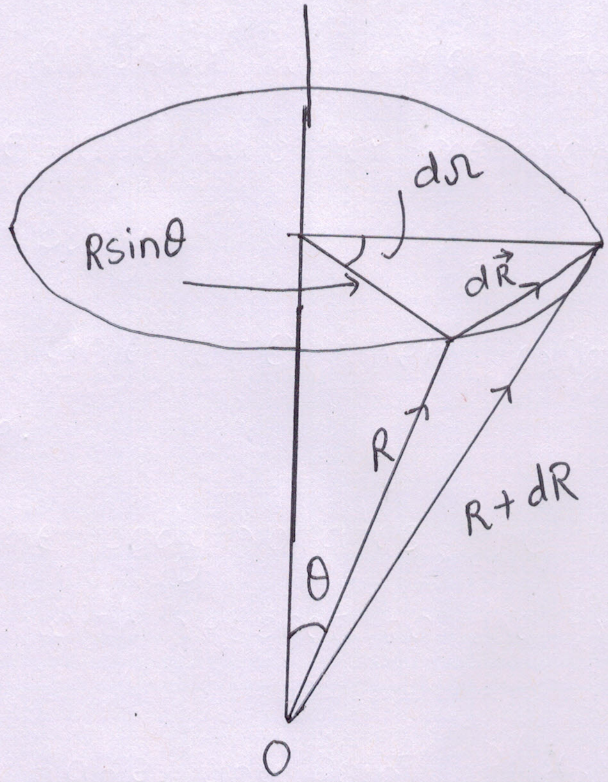
$$\vec{r} = \vec{r}(t)$$

At time $t=0$;

$$\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z \qquad \text{space set}$$

$$= \hat{i}'\,x' + \hat{j}'\,y' + \hat{k}'\,z' \qquad \text{body set}$$

Since OX'Y'Z' system rotates, unit vectors $\hat{i}', \hat{j}', \hat{k}'$ will also change with time whereas unit vectors $\hat{i}, \hat{j}, \hat{k}$ will remain constant.



From fig,

$$dR = R\sin\theta \, d\Omega$$

$$d\vec{R} = d\vec{\Omega} \times \vec{R}$$

$$\frac{d\vec{R}}{dt} = \frac{d\vec{\Omega}}{dt} \times \vec{R}$$

$$\frac{d\vec{R}}{dt} = \vec{\omega} \times \vec{R}$$

so we can write; $\dfrac{d\hat{i}'}{dt} = \vec{\omega} \times \hat{i}'$

where, (ω) is the angular velocity of rotation

The rate of change of $\vec{r}$ can be written as;

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\hat{i}'x' + \hat{j}'y' + \hat{k}'z'\right)$$

$$= x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt} + \hat{i}'\dot{x}' + \hat{j}'\dot{y}' + \hat{k}'\dot{z}'$$

$$= x'(\vec{\omega}\times\hat{i}') + y'(\vec{\omega}\times\hat{j}') + z'(\vec{\omega}\times\hat{k}')$$

$$+ \dot{x}'\hat{i}' + \dot{y}'\hat{j}' + \dot{z}'\hat{k}'$$

$$\left(\frac{d\vec{r}}{dt}\right)_{unprimed} = \vec{\omega}\times\left\{x'\hat{i}' + y'\hat{j}' + z'\hat{k}'\right\} + \left(\frac{d\vec{r}}{dt}\right)_{primed}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{unprimed} = \left(\frac{d\vec{r}}{dt}\right)_{primed} + \vec{\omega}\times\vec{r}$$

In operator form we can write above as;

$$\frac{d}{dt} = \frac{d'}{dt} + \vec{\omega}\times \qquad \text{— (1)}$$

or, $$\frac{d\vec{r}}{dt} = \frac{d'\vec{r}}{dt} + \vec{\omega}\times\vec{r}$$

To ~~th~~ obtain the relationship between accelerations;

$$\frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d'}{dt} + \vec{\omega}\times\right)\left(\frac{d'\vec{r}}{dt} + (\vec{\omega}\times\vec{r})\right)$$

$$= \frac{d'^2\vec{r}}{dt^2} + \left(\frac{d'\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d'\vec{r}}{dt}\right) + \vec{\omega}\times\frac{d'\vec{r}}{dt}$$

$$+ \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d'^2\vec{r}}{dt^2} + 2\vec{\omega}\times\frac{d'\vec{r}}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \cancel{\vec{\omega}\times\frac{d'\vec{r}}{dt}}$$

$$+ \frac{d'\omega}{dt}\times\vec{r}$$

$$\vec{a} = \vec{a'} + 2\vec{\omega}\times\frac{d'\vec{r}}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \frac{d'\vec{\omega}}{dt}\times\vec{r}$$

Take $\frac{d'\omega}{dt} = \frac{d\omega}{dt}$

Then,

$$\boxed{\vec{a} = \vec{a'} + 2\vec{\omega}\times\frac{d'\vec{r}}{dt} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \frac{d\vec{\omega}}{dt}\times\vec{r}} \quad (2)$$

The additional term in eqn (2) appears because of relative rotation of two- co-ordinate systems. Third term $\vec{\omega}\times(\vec{\omega}\times\vec{r})$ is the ~~centrif~~ centripetal acceleration.