B.Sc. Third year: Classical Mechanics:

Chapter 10- Variational Principles and Lagranges's Equations

Lecture 1: *15 July 2020*

Calculus of Variation

Introduction:

Newton's equation of motion can be restated in terms of Lagrange's equations by using d'Alembert Principle. The Euler's Lagrange's equation can be derived by using variation principle called Hamilton's principle. The development of Calculation of variation was started by Newton in 1866 and was extended by Bernoulli, Euler, Legendre, Lagrange, Hamilton and Jacobi.

The problem in calculus of variation is to determine the function $f(x)$ such that the integral ,

$$I = \int_{x_1}^{x_2} f(\, y(x), y'(x), x\,)\, dx \dots \dots \dots (i)$$

Is an extremum i.e. either a minimum or maximum. In above equation $y'(x) = \frac{dy}{dx}$

Let us represent all possible function $y(x)$ by parametric representation,

$y = y(\alpha, x)$ such that when $\alpha = 0, y = y(0, x) = y(x)$

Then we can write;

$$y = y(\alpha, x) = y(0, x) + \alpha\, \eta(x) \dots \dots . (ii)$$

where $\eta(x)$ is some function of x which has a continuous first order derivative and vanishes at the end pointsx_1 and x_2.

Since $y(\alpha, x) = y(x)$ at the end points of the path

$$\eta(x_1) = \eta(x_2) = 0$$

The intergral now becomes a function of α.

For the integral I to have extreme value;

$$\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} = 0$$

Now,

$$\frac{\partial I}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f(y, y', x) dx$$

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left\{\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}\right\} dx$$

Since x is not the function of α, $\frac{\partial x}{\partial \alpha} = 0$

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left\{\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right\} dx$$

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left\{\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial x \partial \alpha}\right\} dx$$

Integrating second term by parts ;

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial \alpha}\right) dx = \frac{\partial f}{\partial y'}\left[\frac{\partial y}{\partial \alpha}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha} dx$$

Also, $\frac{\partial y}{\partial \alpha} = \eta(x)$

$$\left[\frac{\partial y}{\partial \alpha}\right]_{x_1}^{x_2} = \eta(x_2) - \eta(x_1) = 0$$

Thus,

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) \frac{\partial y}{\partial \alpha} \right\} dx$$

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y}\, \eta(x) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) \eta(x) \right\} dx$$

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) \right\} \eta(x)\, dx$$

Since, $\frac{\partial I}{\partial \alpha} = 0$

$$\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

For Reminding;

$$\boldsymbol{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0}$$

Which is the required condition .

Use of Calculus of variation

1. Shortest distance between the two points (Geodesics)

A straight line is regarded as the shortest distance between the two points. Thus straight line is the extremum path of the particle . The equation of such path can be obtained by using technique of calculus of variation.

A small element of arc has the length (ds) and can be represented as;

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{(1 + \dot{y}^2)}\, dx$$

The total length of the curve between two points 1 and 2 can be written as

$$I = \int_1^2 ds = \int_1^2 \sqrt{(1 + \dot{y^2})}\, dx = \int_1^2 f\, dx$$

For this curve to be shortest, $\delta I = 0$ i.e.

$$\boldsymbol{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \ldots\ldots (i)}$$

where $f = \sqrt{(1 + \dot{y^2})}$ must be satisfied.

$$\frac{\partial f}{\partial y} = 0;\ \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{(1 + \dot{y^2})}}$$

Then eq (i) reduces to,

$$\frac{d}{dx}\left[\frac{\dot{y}}{\sqrt{(1 + \dot{y^2})}}\right] = 0$$

$$\frac{\dot{y}}{\sqrt{(1 + \dot{y^2})}} = a; constant$$

from which $\dot{y} = m$

After integration, $y = mx + c$

Which is the equation of straight line.