Variational Principles and Lagrange's Equations

Lecture 3

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Hamilton's Variational Principle

Statement- Hamilton's variational principle for conservative system is stated as follows. The motion of system from time t_1 to t_2 is such that the line integral,

$$I = \int_{t_1}^{t_2} L dt \tag{1}$$

is extremum for the path of motion. where $L = T - V$ is called the Lagrangian.

Let us consider a conservative system of particles. Employing the generalised co-ordinates, the integral can be written as;

$$\int_{t_1}^{t_2} [T(q_j, \dot{q}_j) - V(q_j)] dt \tag{2}$$

Then according to Hamilton's variational principle,

$$\delta \int_{t_1}^{t_2} [T(q_j, \dot{q}_j) - V(q_j)] dt = 0 \tag{3}$$

$$\int_{t_1}^{t_2} \sum_j [(\frac{\partial T}{\partial q_j}\delta q_j + \frac{\partial T}{\partial \dot{q}_j}\delta \dot{q}_j) - \frac{\partial V}{\partial q_j}]dt = 0 \qquad (4)$$

$$\int_{t_1}^{t_2} \sum_j (\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j})\delta q_j dt + \int_{t_1}^{t_2} \sum_j \frac{\partial T}{\partial \dot{q}_j}\delta \dot{q}_j dt = 0 \qquad (5)$$

$$\int_{t_1}^{t_2} \sum_j (\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j})\delta q_j dt + \int_{t_1}^{t_2} \sum_j \frac{\partial T}{\partial \dot{q}_j}\frac{d}{dt}(\delta q_j) dt = 0 \qquad (6)$$

$$\int_{t_1}^{t_2} \sum_j (\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j})\delta q_j dt + \sum_j \frac{\partial T}{\partial \dot{q}_j}\delta q_j \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_j \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j})\delta q_j dt = 0 \qquad (7)$$

In such variation $\delta q_j \Big|_{t_1}^{t_2}$, Hence

$$\int_{t_1}^{t_2} \sum_j (\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j}) \delta q_j dt - \int_{t_1}^{t_2} \sum_j \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) \delta q_j dt = 0 \quad (8)$$

$$\int_{t_1}^{t_2} \sum_j [\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} - \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j})] \delta q_j dt = 0 \quad (9)$$

Since all δq_j are independent of each other the coefficient of δq_j should be zero.

$$\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} - \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) = 0 \quad (10)$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial}{\partial q_j}(T - V) = 0 \quad (11)$$

For conservative system V is not the function of $\dot{q}_j$. Hence

$$\frac{d}{dt}(\frac{\partial}{\partial \dot{q}_j}(T - V)) - \frac{\partial}{\partial q_j}(T - V) = 0 \quad (12)$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = 0 \quad (13)$$

Non-holonomic system (Lagrange's method of undetermined multipliers)

The non-holonomic constraints can be put in the form,

$$\sum_k a_{lk} dq_k + a_{lt} dt = 0, l = 1, 2, 3... \tag{14}$$

The Lagrange's equations are then be obtained by inserting above equation in Hamilton's variational principle. This is the extension of Hamilton's variational principle for non-holonomic systems. This procedure is called Lagrange's method of undetermined multipliers.
The virtual displacement in Hamilton's variational principle are taken at constant time and so above equation becomes,

$$\sum_k a_{lk} dq_k = 0 \tag{15}$$

If there are m constants we have m equations in all $l = 1, 2, 3....m$. Multiply eq (15) by m constants $\lambda_i = \lambda_1, \lambda_2, \lambda_3, ...\lambda_m$ sum over l and integrate resulting equation from point 1 to 2.

$$\int_1^2 \sum_k \sum_l \lambda_l a_{lk} \delta q_k dt = 0 \tag{16}$$

Combine this with Hamilton's principle for conservative systems as below;

$$\delta \int_1^2 L dt = \int_1^2 dt \sum_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right)\right)\delta q_k = 0 \tag{17}$$

$$\int_1^2 dt \sum_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) + \sum_l \lambda_l a_{lk}\right)\delta q_k = 0 \tag{18}$$

All the $q_k' s$ are not independent but are connected by m equations (15). However the first (n-m) of these co-ordinates may be chosen independently the last m ones being fixed by eq (15). Let us choose λ_l's to be such that

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k}) + \sum_l \lambda_l a_{lk} = 0 \tag{19}$$

where $k = n - m + 1, ..., n$ Having found λ_l's from (19) we can write eq(18)

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k}) - \frac{\partial L}{\partial q_k} = \sum_l \lambda_l a_{lk} \tag{20}$$

This is the set of Lagrange's equations for non-holonomic systems.