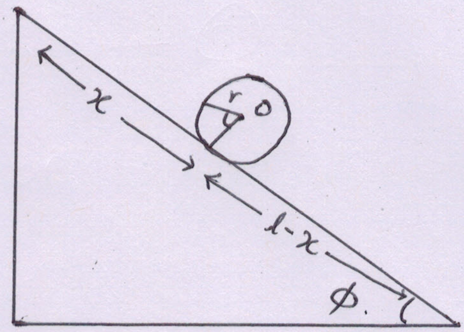
Application of Lagrange's method of undetermined multipliers:

① To find the equation of cylinder rolling down an inclined plane

Let us consider a cylinder rolling down an inclined plane without slipping as shown in figure.



The equation of constraint can be written as,

$$d\theta = \frac{dx}{r}$$

$$r\, d\theta - dx = 0 \quad \text{—} (1)$$

Compairing eq^n with

$$\sum_{K} a_{lK}\, \delta q_K = 0$$

we get, $a_\theta = r \quad ; \quad a_x = -1$

Remember! θ and x are generalized co-ordinates.

Here, KE is made up of two terms i.e. KE of translation plus KE of rotation.

$$\therefore \; KE(T) = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2$$

or, $T = \frac{1}{2} M \dot{x}^2 + \frac{1}{4} M r^2 \dot{\theta}^2$

$\left(\because I = \frac{1}{2} M r^2\right)$

and $PE(V) = mgh$

$(V) = mg\,(l-x)\sin\phi$

where (l) be the length of an inclined plane.

Now, Lagrangian $(L) = T - V$

So, $(L) = \frac{1}{2} M \dot{x}^2 + \frac{1}{4} M r^2 \dot{\theta}^2 - mg(l-x)\sin\phi$ —(2)

The Lagrange's eq^n for 'x' can be written as;

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = a_x \lambda$$

or, $$\frac{d}{dt}\left(M\dot{x}\right) - Mg\sin\phi = -\lambda$$

or, $$M\ddot{x} - Mg\sin\phi + \lambda = 0 \quad \text{—(3)}$$

The Lagrange's equation for 'θ' can be written as;

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = a_\theta \lambda$$

or, $$\frac{d}{dt}\left(\frac{1}{2} M r^2 \dot{\theta}\right) - 0 = r\lambda$$

or, $\frac{1}{2} M r^2 \ddot{\theta} - r\lambda = 0$

or, $\frac{1}{2} M r \ddot{\theta} = \lambda$ —— (4)

Further from eq^n (1), $r\, d\theta = dx$

$$r \frac{d\theta}{dt} = \frac{dx}{dt}$$

$$r \frac{d^2\theta}{dt^2} = \frac{d^2x}{dt^2}$$

$$r\ddot{\theta} = \ddot{x}$$

Putting the value of $r\ddot{\theta}$ in eq^n (4).

or, $\frac{1}{2} M \ddot{x} = \lambda$ —— (5)

Putting value of λ in eq^n (3).

$$M\ddot{x} - Mg\sin\phi + \frac{1}{2} M \ddot{x} = 0$$

$$\frac{3}{2} M \ddot{x} = Mg\sin\phi$$

or, $\ddot{x} = \frac{2}{3} g \sin\phi$

Again, $r\ddot{\theta} = \ddot{x} \Rightarrow \ddot{\theta} = \frac{\ddot{x}}{r}$

$\therefore \ddot{\theta} = \frac{2g\sin\phi}{3r}$

The frictional force of constraint (λ) is given by,

$$\lambda = \frac{1}{2} M \ddot{x}$$

$$\lambda = \frac{1}{2} M \cdot \frac{2}{3} g \sin\phi$$

$$\lambda = \frac{M g \sin\phi}{3}$$

We have,

$$\ddot{x} = \frac{d^2 x}{dt^2}$$

$$\ddot{x} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dx}\left(\frac{dx}{dt}\right) \cdot \frac{dx}{dt}$$

$$\ddot{x} = \frac{dv}{dx} \cdot v$$

$$\ddot{x} = v \frac{dv}{dx}$$

or, $\frac{2}{3} g \sin\phi = v \cdot \frac{dv}{dx}$

or, $v \, dv = \frac{2}{3} g \sin\phi \, dx$

or, $\int_0^v v \, dv = \int_0^l \frac{2}{3} g \sin\phi \, dx$

or, $\frac{v^2}{2} = \frac{2}{3} g \sin\phi \, l$

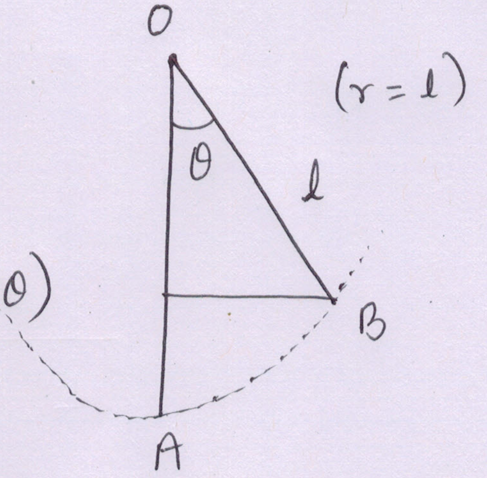
or, $v^2 = \frac{4 g \sin\phi \, l}{3}$

or, $\boxed{v = \left(\frac{4 g \sin\phi \, l}{3} \right)^{1/2}}$

which is the velocity of the cylinder at the bottom.

2. Simple Pendulum

The Lagrangian can be written as;



$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

Here the constraint is,

$$r = l$$

$$r - l = 0$$

So, $a_r = 1$, $a_\theta = 0$ because there is no θ in constraint eqn.

The Lagrange's 'θ' eq is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = a_\theta \lambda$$

$$\frac{d}{dt}\left(m l^2 \dot{\theta}\right) + mgl \sin\theta = 0$$

$$m l^2 \ddot{\theta} + mgl\,\theta = 0 \qquad (\because \sin\theta \simeq \theta)$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

Compairing this with $\ddot{\theta} + \omega^2 \theta = 0$

$$\Rightarrow \omega = \sqrt{\frac{g}{l}}$$

$$T = \frac{2\pi}{\ \ } = 2\pi\sqrt{\frac{l}{\ \ }}$$

3. Particle on a sphere

Let a particle of mass (m) move on the frictionless surface of the sphere of radius (r) under the action of gravity.

Let co-ordinate of any point on the sphere will be (r, θ, ϕ).

If we take that the particle move on a vertical plane on the surface then we can ignore angle ϕ and we can write Lagrangian as;

$$(L) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - mgr\cos\theta \quad \text{—} (1)$$

As the particle move on the surface of sphere the constraint condition is,

$$r = l \text{ (constant)}$$

so, $\quad dr = 0$

So coefficients of constraint are;

$$a_r = 1, \quad a_\theta = 0 \quad \text{—} (2)$$

Further, $\quad \dot{r} = 0 \quad \because r$ is constant

Hence,

$$(L) = \frac{1}{2} m r^2 \dot{\theta}^2 - mgr\cos\theta \quad \text{—} (3)$$

The eq^n of motion in θ will be,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = a_\theta \lambda \quad \text{--- (4)}$$

$$\frac{d}{dt}\left(m r^2 \dot{\theta}\right) - mgr \sin\theta = 0$$

$$m r^2 \ddot{\theta} - mgr \sin\theta = 0 \quad \text{--- (5)}$$

The eq^n of motion in r will be,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = a_r \lambda$$

$$0 - m r \dot{\theta}^2 + mg \cos\theta = \lambda$$

or, $$-m r \dot{\theta}^2 + mg \cos\theta = \lambda$$

This shows Lagrange's multiplier(λ) is dependent on θ.

or, $$-m r \dot{\theta}^2 + mg \cos\theta = \lambda(\theta)$$

differentiating on both sides w.r. to 't'.

or, $$-2 m r \dot{\theta} \ddot{\theta} - mg \sin\theta \, \dot{\theta} = \frac{d(\lambda(\theta))}{dt}$$

or, $$-2 m r \dot{\theta} \ddot{\theta} - mg \sin\theta \, \dot{\theta} = \frac{d\lambda}{d\theta} \frac{d\theta}{dt}$$

or, $$-2 m r \dot{\theta} \ddot{\theta} - mg \sin\theta \, \dot{\theta} = \frac{d\lambda}{d\theta} \dot{\theta}$$

$$-2mr\ddot{\theta} - mg\sin\theta = \frac{d\lambda}{d\theta} \quad \text{—} ⑥$$

From eqⁿ ⑤,

$$\cancel{m} r^2 \ddot{\theta} = \cancel{m} g r \sin\theta$$

$$\Rightarrow \ddot{\theta} = \frac{g r \sin\theta}{r^2}$$

Putting value of $\ddot{\theta}$ in eqⁿ ⑥.

or, $$-2mr.\left(\frac{g r \sin\theta}{r^2}\right) - mg\sin\theta = \frac{d\lambda}{d\theta}$$

or, $$-2mg\sin\theta - mg\sin\theta = \frac{d\lambda}{d\theta}$$

or, $$-3mg\sin\theta = \frac{d\lambda}{d\theta}$$

or, $$\frac{d\lambda}{d\theta} = -3mg\sin\theta$$

Now, integrating

$$\lambda = 3mg\cos\theta + C$$

~~At the top of the sphere $\theta = 0$ so $\lambda = 3mg$~~

The force of constraint at the top of the surface is $\lambda = mg$

so, $\lambda = 3mg\cos\theta + c$

or, $mg = 3mg\cos 0^\circ + c$

or, $mg = 3mg + c$

or, $c = -2mg$

Hence, $\boxed{\lambda(\theta) = 3mg\cos\theta - 2mg}$

For the particle to move on the surface, the force of constraint should be positive i.e.

Surface pushes the particle outward,

$$\lambda(\theta) \geq 0$$

$$3mg\cos\theta - 2mg \geq 0$$

For equaty case, $\cos\theta_c = 2/3$

where (θ_c) is the angle at which the particle flies off the surface.