Conservation Theorems & Symmetry properties

If any system or any function representing a property of the system does not change under some operation, the system is said to possess a symmetry with respect to the given operation. For eg: when a cylinder is rotated about it's axis it's apparent shape does not change. The cylinder is said to have rotational symmetry about it's axis. Such symmetry operations under arbitrary translation give rise to conservation of linear momentum and under arbitrary rotation about arbitrary axis give rise to conservation of angular momentum. Thus symmetries corresponds to conservation laws.

Conservation of angular momentum

If a co-ordinate corresponding to rotation is cyclic, rotation of the system about the given axis has no effect on the description of the system motion i.e. system remains invariant under such co-ordinate rotation and angular momentum is conserved.

If we assume q_j as generalised co-ordinate the T does not depends on q_j and for conservative system (V) does not depends on $\dot{q}_j$

we get, $\dot{p}_j = Q_j$ (generalised force)

If we take q_j as rotation co-ordinate, the generalised force is the component of torque and p_j represents angular momentum.

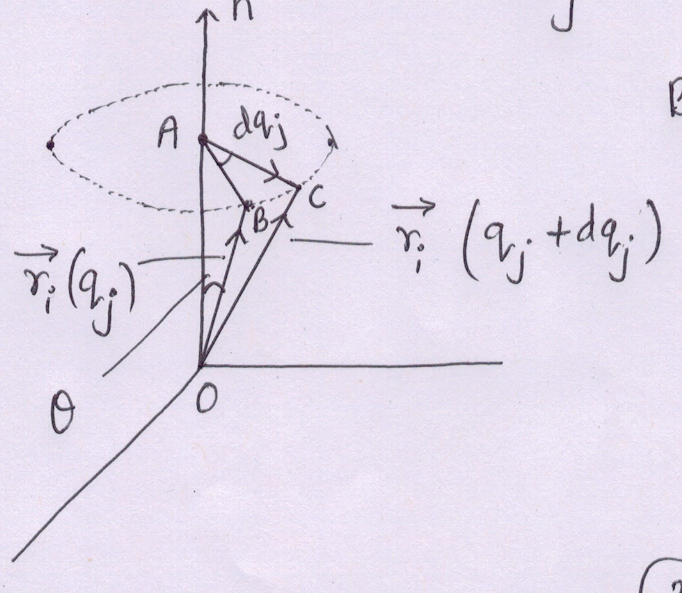
The generalised force, $(Q_j) = \sum_i \vec{F}_i \cdot \dfrac{\partial \vec{r}_i}{\partial q_j}$

From fig,

$|d\vec{r}_i| = AB\, dq_j$

$= r_i \sin\theta\, dq_j$

$$\frac{|\partial \vec{r}_i|}{|\partial q_j|} = r_i \sin\theta$$



$BC = |d\vec{r}_i|$

The direction of which is perpendicular to both $\vec{r}_i$ & $\hat{n}$ is,

$$\frac{\partial \vec{r}_i}{\partial q_j} = \hat{n} \times \vec{r}_i$$

Now,

$$Q_j = \sum_i \vec{F}_i \cdot (\hat{n} \times \vec{r}_i)$$

$$= \sum_i (\hat{n} \times \vec{r}_i) \cdot \vec{F}_i$$

$$= \sum_i \hat{n} \cdot (\vec{r}_i \times \vec{F}_i)$$

$$= \sum_i \hat{n} \cdot \vec{N}_i$$

$$= \hat{n} \cdot \sum_i \vec{N}_i$$

$$Q_j = \hat{n} \cdot \vec{N}$$

where, $\vec{N}$ represents total torque,

Further,

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i v_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$p_j = \sum_i m_i \vec{v}_i \cdot (\hat{n} \times \vec{r}_i)$$

$$p_j = \sum_i \hat{n} \cdot (\vec{r}_i \times m_i \vec{v}_i)$$

$$P_j = \hat{n} \cdot \sum_i \vec{L}_i$$

$$P_j = \hat{n} \cdot \vec{L}$$

Suppose q_j represents cyclic co-ordinate then,

$$Q_j = -\frac{\partial V}{\partial q_j} = 0$$

& hence, $\dot{P}_j = 0$

$$\Rightarrow \quad P_j = \text{constant}$$

$$\boxed{\hat{n} \cdot \vec{L} = \text{constant}}$$

P_j shows the component of angular momentum along axis of rotation.

Conservation of energy

Let us consider that :

(a) a conservative system so that the PE is a function of co-ordinates only and not of velocities.

(b) constraints do not change with time i.e. they are independent of time.

(c) L can be written as: $L(q_j, \dot{q}_j)$

The total time derivative of $L(q_j, \dot{q}_j)$ can be written as;

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} \quad \text{—(1)}$$

Putting, $\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$

Then eq^n (1) becomes,

$$\frac{dL}{dt} = \sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}\right]$$

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right)$$

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial T}{\partial \dot{q}_j} \right)$$

Because for conservative system $\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$

as, $\frac{\partial V}{\partial \dot{q}_j} = 0$.

Also, $\frac{\partial T}{\partial \dot{q}_j} = p_j$

Hence, $\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j p_j \right)$

$$\frac{dL}{dt} - \sum_j \frac{d}{dt}\left(\dot{q}_j p_j \right) = 0$$

$$\frac{d}{dt}\left(\sum_j \dot{q}_j p_j - L \right) = 0$$

$$\Rightarrow \sum_j \dot{q}_j p_j - L = \text{constant} = J \text{ (say)}$$

J is called Jacobi's integral which is nothing but is a Hamiltonian (H).

$$H = \sum_j \dot{q}_j p_j - L = \text{constant}$$

卐 From Euler's Theorem: if f is a homogeneous function of order n, then

$$\sum_j q_j \frac{\partial f}{\partial q_j} = nf$$

But here, $n=2$

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$$

$$\sum_j \dot{q}_j p_j = 2T$$

Thus,

$$H = \sum_j \dot{q}_j p_j - L$$

$$H = 2T - L$$

$$H = 2T - T + V$$

$$H = T + V = KE + PE = \text{constant}$$

Hence total energy is conserved.